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q–Schrödinger Equations for $V = u^2 + 1/u^2$ and Morse Potentials in terms of the q–canonical Transformation

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Abstract

The realizations of the Lie algebra corresponding to the dynamical symmetry group $SO(2,1)$ of the Schrödinger equations for the Morse and the $V = u^2 + 1/u^2$ potentials were known to be related by a canonical transformation. q–deformed analog of this transformation connecting two different realizations of the $sl_q(2)$ algebra is presented. By the virtue of the q–canonical transformation a q–deformed Schrödinger equation for the Morse potential is obtained from the q-deformed $V = u^2 + 1/u^2$ Schrödinger equation. Wave functions and eigenvalues of the q–Schrödinger equations yielding a new definition of the q–Laguerre polynomials are studied.

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1 Introduction

q–harmonic oscillators are the most extensively studied q–deformed systems [1]. However, there is a limited literature on the q–deformations of other systems. In a recent work a new approach was adopted for defining new q–deformed Schrödinger equations in a consistent manner with the q–oscillators [2]: q–deformation of the one dimensional Kepler problem was obtained by the help of q–deformed version of the canonical transformation connecting the x^2 and $1/x$ potentials.

In this work we attempt to study the q–deformation of another problem, namely the Schrödinger equation for the Morse potential. Note that, the Morse potential problem in its undeformed form can already be considered as a kind of q–deformed oscillator [3]. What we are doing here is the q–deformation of the Schrödinger equation for the Morse potential itself. It is the continuation of our program for defining q–deformed Schrödinger equations consistently with the q–oscillators: it has been known for sometime that the Morse potential and the one dimensional oscillator with an extra inverse square potential are connected by a point canonical transformation [4]. Taking the advantage of this fact, we start with the q–Schrödinger equation written for the potential $V = u^2 + 1/u^2;\ u \geq 0$, which is essentially the radial equation for the two dimensional oscillator in polar coordinates. We then introduce a q–canonical transformation relating the above potential to the Morse potential $V = e^{-x} - e^{-2x};\ -\infty < x < \infty$, which enables us to obtain a q–Schrödinger equation for the latter potential.

q–deformation of the Morse potential is also discussed in [5] in terms of ladder and shift operators, where an application to H_2 molecule is given.

Obtaining the solutions of q–Schrödinger equations in general is not easy. For q–harmonic oscillators the problem is somehow less complicated and there can be more than one approach [6]-[8]. For example in [6] the series solution of the undeformed problem is generalized in such manner that the resulting q–deformed recursion relations can be handled. When a similar generalization is performed for the q–deformed Schrödinger equation for $V = u^2 + 1/u^2$ potential one obtains an equation defining the q–Laguerre polynomials which differs from the ones available in the literature [9]. The difference is discussed in section 6.

In section 2 we review the transformation connecting the Schrödinger equations of the $V = u^2 + 1/u^2$ and the Morse potentials. We also present the canonical transformation between the realizations of the $sl(2)$ algebra corresponding to these potentials which both have the same dynamical group $SO(2, 1)$.

In section 3 we introduce the q–deformation of the phase space variables suitable to obtain two realizations of the $sl_q(2)$ algebra without altering the form of the undeformed generators except some overall and operator ordering factors. We

than define the q–canonical transformation connecting these realizations.

In section 4 we first write the q–deformed Schrödinger equation for the potential $V = u^2 + 1/u^2$ which requires a trivial generalization of the q–oscillator potential $V = u^2$. We then employ the q-canonical transformation of Section 3 to arrive at a Schrödinger equation for the q–Morse potential $V = e^{-x} - e^{-2x}$.

In section 5 we attempt to solve the q–Schrödinger equations which define the q–Laguerre polynomials. The general scheme is described and illustrated for the ground and the first exited levels.

2 From the Morse Potential to $V = u^2 + 1/u^2$

Time independent Schrödinger equation for the Morse potential is ($\hbar = 1$)

$$\left(-\frac{1}{2\mu}\frac{d^2}{dx^2} + Ae^{-2x} - Be^{-x} - E^{\mathrm{M}}\right)\phi(x) = 0; \tag{2.1}$$

$x \in (-\infty, \infty)$.

By making use of the variable change [4]

$$x = -2\ln u, \tag{2.2}$$

(2.1) becomes

$$\left(-\frac{1}{2\mu}\frac{d^2}{du^2} + 4Au^2 - \frac{4E^{\mathrm{M}} + 1/8\mu}{u^2} - 4B\right)\psi(u) = 0; \tag{2.3}$$

$u \in [0, \infty)$, with

$$\psi(u) = \frac{1}{\sqrt{u}}\phi(-2\ln u). \tag{2.4}$$

(2.3) is equivalent to the Schrödinger equation for the one dimensional harmonic oscillator with an extra potential barrier

$$V(u) = -\frac{4E^{\mathrm{M}} + 1/8\mu}{u^2} \tag{2.5}$$

with the energy

$$E = 4B. \tag{2.6}$$

The normalized wave functions and the energy spectrum of the Schrödinger equation (2.3) are

$$\psi_n(u) = \sqrt{\frac{2(\mu\omega)^{\nu+1}n!}{\Gamma(\nu+n+1)}}e^{-\frac{\mu\omega}{2}u^2}u^{\nu+\frac{1}{2}}L_n^{(\nu)}(\mu\omega u^2), \tag{2.7}$$

and

$$E = \omega(n + \nu + 1) = 4B; \quad n = 0, 1, 2, \dots, \tag{2.8}$$

where $L_n^{(\nu)}$ are the Laguerre polynomials and $\omega,\ \nu$ are defined as

$$\omega = \sqrt{8A/\mu},\ \nu = -\frac{1}{2}\sqrt{2 + 32\mu E^{\mathrm{M}}}. \tag{2.9}$$

The relation (2.4) and the transformation (2.2) give the solution of the Morse potential Schrödinger equation (2.1) as

$$\phi(x) = \sqrt{\frac{2(\mu\omega)^{\frac{4B}{\omega}-n} n!}{\Gamma(4B/\omega)}} \exp(-\frac{\mu\omega}{2} e^{-x}) \exp[(-\frac{4B}{\omega} - n - 1)x] L_n^{(\frac{4B}{\omega}-n-1)}(\mu\omega e^{-x}), \tag{2.10}$$

where by the virtue of (2.8) $4B/\omega$ has been replaced by $n + \nu + 1$. To obtain the energy spectrum we insert (2.9) into (2.8) and solve for E^{M} :

$$E^{\mathrm{M}} = \frac{1}{8\mu}\left[(\frac{4B}{\omega} - n - 1)^2 - 1/2\right]. \tag{2.11}$$

Dynamical symmetry group of the above systems is $SO(2,1)$. For the Morse potential the phase space realization of the related Lie algebra $sl(2)$ is given by

$$\begin{aligned} M_0 &= -2p - i, \\ M_+ &= -\frac{1}{2}e^{-x}, \\ M_- &= -2p^2 e^x + \alpha e^x. \end{aligned} \tag{2.12}$$

The realization relevant to $V = u^2 + 1/u^2$ potential is

$$\begin{aligned} L_0 &= up_u + \frac{i}{2}, \\ L_+ &= -\frac{1}{2}u^2, \\ L_- &= -\frac{1}{2}p_u^2 + \frac{\alpha}{u^2}. \end{aligned} \tag{2.13}$$

In terms of the usual relations between the coordinates and momenta

$$px - xp = i,\ p_u u - up_u = i,$$

commutation relations satisfied by the above generators can be found to be

$$[M_0, M_\pm] = \pm 2iM_\pm, \quad [M_+, M_-] = -iM_0, \tag{2.14}$$

$$[L_0, L_\pm] = \pm 2iL_\pm, \quad [L_+, L_-] = -iL_0. \tag{2.15}$$

Eigenvalue equation for the Morse potential hamiltonian of (2.1)

$$\left(\frac{1}{2}p^2e^x + Ae^{-x} - E^M e^x\right) e^{-x}\phi x = Be^{-x}\phi(x), \tag{2.16}$$

is equivalent to

$$\left(\frac{-1}{4\mu}M_- - 2AM_+\right)\xi(x) = B\xi(x), \tag{2.17}$$

with the parameter α and the eigenfunction ξ are identified as

$$\alpha \equiv 4\mu E^M;\ \xi(x) \equiv e^{-x}\phi(x). \tag{2.18}$$

Similarly $\beta u^2 + \gamma/u^2$ potential problem can equivalently be written as the eigenvalue equation

$$\left(\frac{-1}{\mu}L_- - 2\beta L_+\right)\psi = E\psi, \tag{2.19}$$

with the identification

$$\alpha \equiv -\mu\gamma. \tag{2.20}$$

Note that the two identifications of α given in (2.18) and (2.20) are consistent with (2.5). The $1/8\mu u^2$ term is the operator ordering contribution resulting from the point canonical transformation in the Schrödinger equation. The algebraic equations (2.17) and (2.19) are free of the operator ordering term.

Classically, $sl(2)$ algebra realizations relevant to the Morse and the $V = u^2 + 1/u^2$ potentials are connected by the canonical transformation

$$u = e^{-x/2},\ p_u = -2e^{x/2}p. \tag{2.21}$$

Indeed this canonical transformation suggests (2.2). Note that the constant terms in M_0 and L_0 which result from the operator ordering, are not the same and they drop in the classical limit.

3 q–canonical transformation

q–deformation of the Schrödinger equation for the potential $V = u^2 + 1/u^2$ can be introduced similarly to the q–harmonic oscillator. Then, by introducing the q–canonical transformation suggested by (2.21) we can arrive at a q–deformed Morse potential Schrödinger equation.

We introduce q–deformation in terms of q–deformed commutation relations between the phase space variables p, x and p_u, u. Note that we use the same notation for the undeformed and q–deformed objects.

There are more than one definitions of q–canonical transformations. Most of them were introduced in terms of some properties of the basic q-commutators [10]. However, they are not suitable for our purpose.

Let us recall the definition of q–canonical transformations given in [2]: we keep the phase space realizations of the q–deformed generators of the dynamical symmetry group to be formally the same as the undeformed generators $M_i(x,p)$ and $L_i(u,p_u)$, up to operator ordering corrections and overall factors. We then define the transformation $x,p \to u,p_u$ to be the q–canonical transformation if
$i)$ q–algebras generated by $M_i(x,p)$ and $L_i(u,p_u)$ are the same and,
$ii)$ the q–commutation relations (dictated by the first condition) between p and x, and p_u and u are preserved.

Let the q–deformed phase space variables satisfy ($\hbar = 1$)

$$p_u u - q u p_u = i. \tag{3.1}$$

We then define the q–deformation of the generators of (2.13) as

$$\begin{aligned} L_0 &= K^{-1}(up_u + ic). \\ L_+ &= -K^{-1/2}\frac{\sqrt{q}}{1+q}u^2, \\ L_- &= K^{-1/2}(-\frac{\sqrt{q}}{1+q}p_u^2 + \frac{\alpha}{u^2}), \end{aligned} \tag{3.2}$$

where the constants are

$$c = \frac{1}{q(1+q)} - \frac{(q-1)(q^2+1)}{q\sqrt{q}}\alpha,$$
$$K = \frac{1}{q^2\sqrt{q}}(1+q^2)\left(\sqrt{q} + (1-q)(1-q^2)\alpha\right).$$

The generators (3.2) satisfy

$$\begin{aligned} L_0L_- - \frac{1}{q^2}L_-L_0 &= -\frac{i}{q}L_-, \\ L_0L_+ - q^2L_+L_0 &= iqL_+, \\ L_+L_- - \frac{1}{q^4}L_-L_+ &= -\frac{i}{q^2}L_0, \end{aligned} \tag{3.3}$$

which is the $sl_q(2)$ algebra introduced in [11].

To obtain another realization of the $sl_q(2)$ algebra let us define the q–commutator of the q–deformed variables p and x to be

$$pe^{-x/2} - qe^{-x/2}p = -\frac{i}{2}e^{-x/2}. \tag{3.4}$$

We introduce the q–deformation of the generators (2.12) as

$$\begin{aligned} M_0 &= F^{-1}(-2p+ib), \\ M_+ &= -\left[\frac{F(q+1)}{2q^2}\right]^{-1/2} e^{-x}, \\ M_- &= \left[\frac{F}{2q(q+1)}\right]^{-1/2} (-\frac{\sqrt{q}}{1+q}p^2e^x + \alpha e^x), \end{aligned} \tag{3.5}$$

where

$$b = -\frac{q+1}{2} + \frac{2\alpha}{\sqrt{q}}(1-q^4),$$
$$F = 1 + \frac{b(1-q^2)+1}{q}.$$

The realization given by (3.5) satisfies the $sl_q(2)$ algebra

$$\begin{aligned} M_0M_- - \frac{1}{q^2}M_-M_0 &= -\frac{i}{q}M_-, \\ M_0M_+ - q^2M_+M_0 &= iqM_+, \\ M_+M_- - \frac{1}{q^4}M_-M_+ &= -\frac{i}{q^2}M_0, \end{aligned} \tag{3.6}$$

which is the same as (3.3).

It is obvious that both the relation between (3.2) and (3.5); and the relation between the q–commutators (3.1) and (3.4) are given by the transformation (2.21). Since both of the q-deformed dynamical systems satisfy the same $sl_q(2)$ algebra (3.3), (3.6), we conclude that the transformation (2.21) is the q–canonical transformation between two systems.

4 q–Schrödinger equations

Once the q–canonical transformation relating the q–dynamical systems given by (3.2) and (3.5) is found, we may proceed in the opposite direction presented in Section 2: first define q–Schrödinger equation for the $V = 1/u^2 + u^2$ potential and then adopt a change of variable suggested by the q–canonical transformation (2.21) to define a q–Schrödinger equation for the Morse potential. To this end we define the q-deformed Schrödinger equation

$$(-\frac{1}{2\mu}D_q^2(u) + A_qu^2 + \frac{\alpha_q}{u^2} - E_q)\psi_q(u) = 0, \tag{4.1}$$

by using the q–hamiltonian

$$H_q = \frac{-1}{\mu} L_- - \mu A_q L_+, \tag{4.2}$$

where we used the realization $p_u = iD_q(u)$ and

$$\alpha_q \equiv -\frac{q^{-1/2} + q^{1/2}}{2\mu}\alpha. \tag{4.3}$$

To reproduce the undeformed equation (2.3) and the definitions of (2.9) in the $q = 1$ limit, the constants A_q and α_q should be chosen to satisfy

$$A_{q=1} = \frac{1}{2}\omega^2 \mu,$$
$$\alpha_{q=1} = \frac{-1}{2\mu}(\nu^2 - 1/4).$$

$D_q(u)$ is the q-deformed derivative defined as [12]

$$D_q(u) f(u) \equiv \frac{f(u) - f(qu)}{(1-q)u}. \tag{4.4}$$

The variable change suggested by (2.21)

$$u = \exp(-x/2), \tag{4.5}$$

leads to the q-deformed Schrödinger equation for the Morse potential

$$(-\frac{1}{2\mu}\mathcal{D}_q^2(x) + A_q e^{-2x} - E_q e^{-x} + \alpha_q)\phi_q(x) = 0. \tag{4.6}$$

Here the wave function ϕ_q is given by

$$\phi_q(x) = e^x \psi_q(e^{-x/2}), \tag{4.7}$$

and the coefficient α_q is identified with the energy of the q–Schrödinger equation for the Morse potential

$$E_q^{\mathrm{M}} \equiv \alpha_q. \tag{4.8}$$

The kinetic term is defined as

$$\mathcal{D}_q^2(x) \equiv \frac{1}{(1-q)^2}[1 - \frac{1+q}{q} e^{2\ln q} e^{-2\ln q \partial_x} + \frac{1}{q} e^{4\ln q} e^{-4\ln q \partial_x}], \tag{4.9}$$

which in the $q=1$ limit becomes

$$\lim_{q\to 1}\mathcal{D}_q^2(x)=\frac{\partial^2}{\partial x^2}.$$

Obviously, the kinetic term (4.9) is an unusual one. The origin of this fact lies in the observation that the undeformed Morse potential Schrödinger equation itself can be viewed as a deformed object whose deformation parameter is the scale of x [3].

Similar to the nondeformed case, by using the q–hamiltonian

$$H_q^M \equiv \frac{-1}{4\mu}M_- - A_q M_+ \tag{4.10}$$

where the realization $p=i\mathcal{D}$ and the identification of α as given in (4.3) the q–Schrödinger equation (4.6) can be written as an algebraic eigenvalue equation

$$H\psi_q = B\psi_q. \tag{4.11}$$

5 Solutions of the q-Schrödinger equations

Inspired by the form of the undeformed solution (2.7), we try to build the solutions of (4.1) on the ground state of the q–Schrödinger equation of the harmonic oscillator (obtained by setting $\alpha_q=0$ in (4.1)) given in terms of the q–exponential [2]

$$e_q(z^2)=1+\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\frac{2(1-q)}{1-q^{2k}}\right)z^{2n}, \tag{5.1}$$

which is defined to satisfy

$$D_q(z)e_q(z^2)=2ze_q(z^2).$$

In terms of the constants μ, ω and $\tilde{\nu}$ which can be identified, respectively, as the mass, frequency of the harmonic oscillator and the generalization of ν, let us choose

$$A_q = \frac{\mu}{2}q^{2\tilde{\nu}+2}\omega'^2, \tag{5.2}$$

$$\alpha_q = \frac{1}{2\mu}[\tilde{\nu}+1/2]_q[\tilde{\nu}-1/2]_q, \tag{5.3}$$

where

$$\omega'=\frac{q^{-\tilde{\nu}+1/2}[\omega]_q[2\tilde{\nu}+2]_q}{[\tilde{\nu}+3/2]_q q+[\tilde{\nu}+1/2]_q}, \tag{5.4}$$

and

$$[O]_q \equiv \frac{1-q^O}{1-q}. \tag{5.5}$$

One can easily verify that the wave function

$$\psi_{0,q}(u) = u^{\tilde{\nu}+1/2} e_q(-\mu\omega' u^2/2), \tag{5.6}$$

is the solution of (4.1) with the energy

$$E_{0,q} = \frac{[\omega]_q}{2}[2\tilde{\nu}+2]_q. \tag{5.7}$$

Since (5.6) and (5.7) are the generalizations of the ground state wave function and energy of the undeformed Schrödinger equation, we identify $\psi_{0,q}(u)$ as the ground state of the q–Schrödinger equation (4.1).

We then start to build the other solutions on the ground state (5.6). Introduce the *Ansatz* for the n*th* state

$$\psi_{n,q} \equiv L_{n,q}(u)\psi_0(u, q^n), \tag{5.8}$$

where

$$\psi_0(u, q^n) \equiv u^{\tilde{\nu}+1/2} e_q(\frac{-\mu\omega' u^2}{2q^n}). \tag{5.9}$$

Substituting (5.8) into (4.1) we obtain

$$\begin{aligned} -\frac{1}{2\mu}D_q^2(u)L_{n,q}(u) + [2]_q q^{-1}\left(\frac{\omega'}{2}q^{\tilde{\nu}-n+1/2}u - \frac{1}{2\mu}[\tilde{\nu}+1/2]_q u^{-1}\right) D_q(u)L_{n,q}(qu) \\ +A_q u^2\left(L_{n,q}(u) - q^{-2n}L_{n,q}(q^2u)\right) + \frac{\alpha_q}{u^2}\left(L_{n,q}(u) - L_{n,q}(q^2u)\right) \\ +q^{-n}E_{0,q}L_{n,q}(q^2u) - E_{n,q}L_{n,q}(u) = 0. \end{aligned} \tag{5.10}$$

Since we look for solutions possessing the correct $q = 1$ limit, we only deal with $n =$ even eigenfunctions and write

$$L_{n,q}(u) = \sum_{k=0}^{n/2} a_{2k}^{(n)} u^{2k}. \tag{5.11}$$

Inserting the above polynomial into (5.10) we arrive at the three term recursion relations

$$\mathcal{P}_{2k}^{(n)} a_{2k}^{(n)} + \mathcal{R}_{2k}^{(n)} a_{2k-2}^{(n)} + \mathcal{Q}_{2k}^{(n)} a_{2k-4}^{(n)} = 0; \ 1 \le k \le n/2+1, \tag{5.12}$$

where

$$\mathcal{Q}_2^{(n)} = \mathcal{P}_{n+2}^{(n)} = 0, \tag{5.13}$$

and

$$\mathcal{P}_{2k}^{(n)} = -\frac{1}{2\mu}[2k]_q[2k-1]_q - \frac{[2]_q}{2\mu q}[\tilde{\nu}+1/2]_q[2k]_q q^{2k} + \alpha_q(1-q^{4k}), \tag{5.14}$$

$$\mathcal{R}_{2k}^{(n)} = \frac{\omega'}{2}[2]_q q^{-(\tilde{\nu}+n-2k+3/2)}[2k-2]_q + q^{-n+4k-4}E_{0,q} - E_{n,q}, \tag{5.15}$$

$$\mathcal{Q}_{2k}^{(n)} = A_q(1-q^{-2n+4k-8}). \tag{5.16}$$

(5.12) can be transformed into the two term recursion relations

$$\mathcal{S}_{2l}^{1}(n)a_{2l}^{(n)} + \mathcal{S}_{2l}^{2}(n)a_{2l-2}^{(n)} = 0;\ 1 \le l \le n/2+1, \tag{5.17}$$

and an $(n/2+1)th$ order equation for the energy eigenvalue $E_{n,q}$

$$\sum_{k=0}^{n/2+1} f_k(n,q)E_{n,q}^k = 0. \tag{5.18}$$

Here $\mathcal{S}_{2l}^{1,2}(n)$, the coefficients $f_k(n,q)$ and then the energy $E_{n,q}$ should be found for each n. To be sure that the energy eigenvalues $E_{n,q}$ lead to the correct $q=1$ limit, we define

$$E_{n,q} = \frac{[\omega]_q}{2}[2n+2\tilde{\nu}+2]_q + K_{n,q}, \tag{5.19}$$

where $K_{n,q}$ will be found as the solution of (5.18) subject to the condition

$$K_{n,q=1} = 0.$$

To have an insight, let us deal with $n=2$ case:
Solution of the q–Schrödinger equation (4.1) is

$$\psi_{2,q} = (a_0^{(2)} + a_2^{(2)}u^2)u^{\tilde{\nu}+1/2}e_q(-\frac{\mu\omega'}{2q}u^2), \tag{5.20}$$

whose coefficients satisfy

$$\{[2]_q + q[2]_q^2[\tilde{\nu}+1/2]_q + 2\mu\alpha_q(q^4-1)\}a_2^{(2)} + \{\mu[\omega]_q q^{2\tilde{\nu}}[4]_q + 2\mu K_{2,q}\}a_0^{(2)} = 0, \tag{5.21}$$

$$\{\omega'[2]_q^2 q^{-\tilde{\nu}+1/2} - [\omega]_q q^{-2}[4]_q - 2K_{2,q}\}a_2^{(2)} + 2A_q(1-q^{-4})a_0^{(2)} = 0. \tag{5.22}$$

We can choose $a_2^{(2)} = 1$, then (5.21) leads to

$$a_0^{(2)} = \frac{\{[2]_q + q[2]_q^2[\tilde{\nu} + 1/2]_q + 2\mu\alpha_q(q^4 - 1)\}}{\mu[\omega]_q q^{2\tilde{\nu}}[4]_q + 2\mu K_{2,q}}, \tag{5.23}$$

where due to (5.22) $K_{2,q}$ satisfies the second order equation

$$K_{2,q}^2 + \alpha(q) K_{2,q} + \beta(q) = 0, \tag{5.24}$$

with the coefficients given by

$$\alpha(q) = \frac{[\omega]_q}{2}\{\frac{[2]_q q^2[-2\tilde{\nu} - 1]_q}{[\tilde{\nu} + 3/2]_q[\tilde{\nu} + 1/2]_q} + [4]_q(q^{-2} + q^{2\tilde{\nu}})\},$$
$$\beta(q) = \frac{-1}{4\mu} A_q(1 - q^{-4}\{[2]_q + q[2]_q^2[\tilde{\nu} + 1/2]_q + 2\mu\alpha_q(q^4 - 1)\}). \tag{5.25}$$

Only one of the two solutions of (5.24) satisfy $K_{2,q=1} = 0$ condition:

$$K_{2,q} = \frac{-1}{2}\alpha(q) + \frac{1}{2}\sqrt{\alpha^2(q) - 4\beta(q)}. \tag{5.26}$$

The calculations for the higher states can be carried out in a straightforward manner in spite of their messy character.

To obtain the energy eigenvalues of the q-Morse problem one should solve $\tilde{\nu}$ in terms of $E_{n,q}$ from (5.19) as $\tilde{\nu}_{n,q} = \tilde{\nu}(n, \omega, q, E_{n,q})$, with $E_{n,q}$ now playing the role of the coefficient of the q–Morse potential. Inserting this into (5.3) and using the identification (4.8) one arrives at

$$E_{n,q}^{\mathrm{M}} = \frac{1}{2\mu}[\tilde{\nu}(n, \omega, q, E_{n,q}) + 1/2]_q[\tilde{\nu}(n, \omega, q, E_{n,q}) - 1/2]_q.$$

Obviously, the wave functions which are the solutions of the q–Schrödinger equation for the Morse potential (4.6) corresponding to the above energy eigenvalues are given by

$$\phi_{n,q}(x) = e^x \psi_{n,q}(e^{-x/2}). \tag{5.27}$$

As an illustration it is enough to present the formulas for the ground state: $\tilde{\nu}_{0,q}$ can be solved as

$$\tilde{\nu}(0, \omega, q, E_{0,q}) = \frac{\ln\{1 - \frac{2(1-q)}{[\omega]_q}E_{0,q}\}}{2\ln q} - 1. \tag{5.28}$$

Hence the q–Schrödinger equation (4.6) possesses the ground state solution

$$\phi_{0,q}(x) = e^x \exp\left(\frac{-x}{4\ln q}\ln\{1-\frac{2(1-q)}{[\omega]_q}E_{0,q}\} + \frac{x}{2} + \frac{1}{4}\right) e_q\left(-\frac{\mu\omega'}{2}e^{-x}\right) \quad (5.29)$$

with the energy

$$E^{\mathrm{M}}_{0,q} = \frac{1}{2\mu}\left[\frac{\ln\{1-\frac{2(1-q)}{[\omega]_q}E_{0,q}\}}{2\ln q} - \frac{1}{2}\right]_q \left[\frac{\ln\{1-\frac{2(1-q)}{[\omega]_q}E_{0,q}\}}{2\ln q} - \frac{3}{2}\right]_q . \quad (5.30)$$

6 Discussions

Application of a q–canonical transformation enabled us to obtain q–Morse potential consistent with the deformed oscillator like potential $V = u^2 + 1/u^2$. Polynomial solutions of the q–Schrödinger equation of the latter (4.1) lead to a new definition of q–Laguerre polynomials (5.10). (4.1) is an eigenvalue equation which does not involve the first q–derivatives. The other definitions of q–Laguerre polynomials [9] can be shown to lead after a suitable coordinate change and wave function *Ansatz* to equations which are essentially of the form

$$D_q^2(u)\phi(u) + \nu(u)\phi(qu) - c_q\phi(u) = 0, \quad (6.1)$$

where c_q is a constant which can be vanishing.

Obviously (6.1) leads to the Schrödinger equation in $q = 1$ limit, but it is not an eigenvalue equation as in (4.1): the scaling operator is equivalent to the first order q–derivative.

Of course, one can always add some terms which are vanishing in $q = 1$ limit to the q–deformed objects without altering the non-deformed limit. Thus, if we permit the appearance of scaling operator in q–Schrödinger equation there can be infintely many varieties. In our approach however there is only one possible definition of q–Schrödinger equation.

Orthogonality properties of the q–Laguerre polynomials hence the inner product of the related Hilbert space should be studied to discuss the hermiticity properties of the q–deformed objects.

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